V
Systems of Paraconsistent Logic

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1. Paraconsistency: characterization and motivation

Let $\models$ be a relation of logical consequence. $\models$ may be defined either semantically ($\Sigma \models A$ holds iff for some specified set of valuations, whenever all the formulas in Σ are true under an evaluation, so is A) or proof theoretically ($\Sigma \models A$ holds iff for some specified set of rules, there is a derivation of A, all of whose (undischarged) premisses are in Σ), or in some other way. $\models$ is *explosive* iff for all A and B, $\{A, {\sim}A\} \models B$. It is *paraconsistent* iff it is not explosive. A logic is *paraconsistent* iff its logical consequence relation is. If a logic is defined in terms of a set of theses it may have more than one associated consequence relation. For example, $\{A_1 \ldots A_n\} \models B$ iff $\vdash (A_1 \wedge \ldots \wedge A_n) \rightarrow B$ or $\vdash A_1 \rightarrow (\ldots \rightarrow (A_n \rightarrow B) \ldots)$ or $A_1, \ldots, A_n \rightarrow B$ (the last representing the theorem-preserving or *weak* inferential connection). In this case all its associated consequence relations should be paraconsistent.

Let Σ be a set of statements. Σ is *inconsistent* iff, for some A, $\{A, {\sim}A\} \subseteq \Sigma$. Σ is *trivial* iff for all B, $B \in \Sigma$. The important fact about paraconsistent logics is that they provide the basis for inconsistent but non-trivial theories. In other words, there are sets of statements closed under logical consequence which are inconsistent but non-trivial. This fact is sometimes taken as an alternative definition of 'paraconsistent' and, given that logical consequence is transitive, it is equivalent to the original definition. The proof is this: If Σ is an inconsistent but non-trivial theory then obviously the consequence relation is paraconsistent. Conversely, suppose that $\{A, {\sim}A\} \not\models B$. Let Σ be the transitive closure of $\{A, {\sim}A\}$ under logical consequence. Then Σ is inconsistent but $B \notin \Sigma$. Because of the equivalence we also call any inconsistent but non-trivial theory *paraconsistent*, and derivatively, any position whose deductive closure provides a paraconsistent theory.

Why should one be interested in paraconsistent logics? Among the many reasons are proof theoretic and semantic ones.

1.1. The proof theoretic reason

The proof theoretic reason is that there are interesting theories T which are inconsistent but non-trivial. Clearly the underlying logic of such theories must be paraconsistent—hence the need to study paraconsistent logics. Examples of inconsistent but non-trivial theories are easy to produce, and many will be given in what follows. A first example, that will recur again and again, is naive set theory, the theory of sets based on the full abstraction axiom scheme, $\exists y \forall x (x \in y \leftrightarrow A)$. This, together with extensionality, characterizes the intuitive conception of set. The theory is inconsistent since it generates the set theoretic paradoxes (e.g. where R is the Russell set, defined as $\{x: \sim x \in x\}$, standard paradox arguments show that $R \in R$ and $\sim R \in R$). Yet it is non-trivial because there are many claims about sets which the intuitive notion rightly rejects (e.g. that $\{\Lambda\} \in \Lambda$, where Λ is the null set). A very similar, and likewise important example, is naive semantics, the truth theory based on the full T-scheme, $Tr\ulcorner A \urcorner \leftrightarrow A$. This characterizes the intuitive conception of truth. It is inconsistent because it generates the semantic paradoxes (e.g., Liar paradoxes). Yet it is non-trivial since there are many claims concerning truth which the naive notion rightly rejects (e.g. that $Tr\ulcorner A \vee B \urcorner \leftrightarrow Tr\ulcorner A \urcorner \wedge Tr\ulcorner B \urcorner$).

Another group of examples of inconsistent but non-trivial theories derive from the history of science. Consider, for example, the Newton–Leibniz versions of the calculus. Let us concentrate on the Leibniz version. This was inconsistent since it required division by infinitesimals. Hence if α is any infinitesimal, $\alpha \neq 0$. Yet it also required that infinitesimals and their products be neglected in the final value of the derivative. Thus $\alpha = 0$. (As much was pointed out by Berkeley in his critique of the calculus.[1]) Despite this the calculus was certainly non-trivial. None of Newton, Leibniz, the Bernoullis, Euler, and so on, would have accepted that $\int_0^1 x\,dx = \pi$. A very different but most interesting example of an inconsistent but non-trivial theory in the history of the natural sciences is the Bohr theory of the atom.[2] According to this an electron could orbit the nucleus of an atom without radiating energy. However, according to Maxwell's equations which formed an integral part of Bohr's account of the behaviour of the atom, an accelerating electron, such as an electron in orbit, must radiate energy. Despite this the Bohr theory of the atom was non-trivial. Someone who suggested to Bohr that it followed from his theory that electrons moved in squares would, rightly, have received a sharp answer. Many other examples of inconsistent but non-trivial theories from the history of science could be given.[3] Indeed it could be persuasively argued that the whole state of scientific knowledge at any time is such a theory.[4] However these two examples will suffice for present illustrative purposes.

A third group of examples of inconsistent but non-trivial theories comprises certain bodies of information which are theories only in a somewhat attenuated sense. What justifies their inclusion in the present setting is that inferences are made, and made commonly, from the information. Thus ideally they may be conceived of as deductively closed corpuses or theories. Many examples could be given here, and will be introduced subsequently.[5]

Among the more interesting non-philosophical examples are certain bodies of law, such as bills of rights and constitutions. The following is a convenient hypothetical example which, however, makes the point clearly. The constitution of a certain country contains the clauses (a) 'No person of the female sex shall have the right to vote', (b) 'All property holders shall have the right to vote'. We may also suppose that it is part of the common law that women may not legally be property holders. As enlightenment creeps over the country this part of common law is changed to allow women to hold property. Inevitably, eventually, a woman, call her Jan, turns up at a polling booth claiming the right to vote. A test case ensues. Patently the law is inconsistent. According to the law Jan both does and does not have the right to vote. Patently, also the law is not trivial. Someone who argued that her cat should be allowed to vote on the basis of (a) and (b) would not get very far. Actual historical examples of inconsistent legal situations are of course more complex and, therefore, more controversial. However two actual examples are the case of Riggs v Palmer and Lincoln's Proclamation of Emancipation. In the former the clear legal right of inheritance was contradicted by the legal principle that no one shall acquire property by crime. The benefactor had, in fact, murdered the deceased. In the second, the freeing of slaves, who were undoubtedly legal property, with no compensation, contradicted the Fifth Amendment, which says that property shall not be taken without just compensation.[6]

Other examples of inconsistent information from which inferences are drawn include: the data presented to a jury in a trial; the information fed into a computer; a person's set of beliefs.[7] In each of these cases the information may obviously be inconsistent. Moreover, inferences are obviously made from this information. Yet clearly people are *not* at liberty to conclude anything they like from the information. That there are inconsistent but non-trivial theories is thus well established.

1.2. The semantical reason

A second reason for being interested in paraconsistent logics is the fact that there are true contradictions, that is, there are statements A and $\sim$A such that both are true. Because of this, some inferences of the form A, $\sim$A/B must fail to be truth-preserving (let alone valid) since some statements (take one such for B) are not true. Thus, Logic is paraconsistent.

Examples of the alleged true contradictions are not difficult to provide. Under the influence of Zeno's paradoxes, Hegel thought that a moving object realized a contradiction: a body in motion was both at a certain place at a certain time and not at it.[8] However, the validity of Zeno's arguments is decidedly doubtful.[9] Hence such dialectic examples of true contradictions are perhaps not so plausible. Much more persuasive examples of true contradictions are provided by the logical paradoxes. These are examples of arguments in set theory and semantics which appear to be perfectly sound arguments issuing in contradictory conclusions. If this is indeed the case then clearly the contradictory conclusions are true. Those who wish to deny this conclusion must show that the paradoxical arguments are not *really* sound at all. This poses the problem of where to locate the unsoundness. It is some measure of the unworkability of the unsoundness position that there is still absolutely no consensus as to where to locate the unsoundness (as there is, for example, with Zeno's paradoxes) and this some 2,000 years after the initial discovery of a logical paradox.

But how is it possible for a contradiction to be true? Quite simply. For example consider the sentence

(c) This is a false sentence of English.

This has two components, a subject 'this' and a predicate 'is a false sentence of English'. Each of the components has certain semantic conditions. Thus, the semantic condition of 'this' is its referring to a certain object—in this case, (c) itself. The semantic condition of the predicate is that it applies truly to a certain class of objects, viz. those which are false English sentences. Now of course (c) is contradictory. In other words the semantic conditions of the components of (c) *overdetermine* its true value. They determine it to be both true and false. Of course, one can state dogmatically that this is incorrect, that we have got the semantic conditions of the components wrong or something of this kind. But this is to elevate consistency into an inviolable constraint on semantics; and why should we suppose it is? Semantic conditions were not laid down by God or even by some Hilbert who kibitzed them for consistency before unleashing them on the world. They have grown up in a piecemeal and haphazard way. It would, quite frankly, be amazing if they *were* consistent. Semantic conditions can be seen as determining a field of meaning. Overdetermining truth conditions produce singularities and other discontinuities in the field. But such are to be expected, and in no way interfere with the rest of the field. Of course this is only a metaphor, but it can help to break a mental set.

Once one gets past this mental block—past the consistency hang-up—there are other plausible examples of true contradictions. For example, in the hypothetical legal set up described in the previous section it seems that 'Jan has the legal right to vote' and 'Jan does not have the legal right to vote' are both true. A somewhat more controversial example[10] concerns the application of multicriterial terms. For instance, to determine whether a phrase, such as 'below 0°C', correctly applies to a certain situation, we may observe the behaviour of either a correctly functioning alcohol thermometer or a correctly functioning thermo-electric thermometer. These work on quite different principles, and there is no sense in which one is more basic to our determination of, or understanding of, temperature than the other. Certain behaviour of either of these instruments provides a sufficient condition for the correct applicability of the term 'below 0°C' or its negation, and both have equal claim to determine an operational meaning of the phrase. Normally the world is such that these two criteria hold or fail together. However, in a novel situation they may well fall apart. In such a situation both the assertion that the phrase applies and the assertion that its negation does are true. By the symmetry of the situation neither claim can be truer than the other. Hence either both are true or both are false. To suppose that both are false would be to deny that they were criteria in the first place. Thus they must both be true. An historical example of where criteria fell apart in this way is the Michaelson-Morley experiment. Because of rigid rod measurements, 'The arms of the Michaelson-Morley interferometer are congruent' was true. Because of measurement in terms of time taken by light rays, 'The arms of the Michaelson-Morley interferometer are not congruent' was true.[11]

Clearly there is a relationship between the proof-theoretic and the semantic motivations for paraconsistency. If the semantic rationale is correct, then the proof theoretic one is too. For if S is the set of things true in some domain containing true contradictions then S is an inconsistent but non-trivial theory. However, it is possible to accept the proof theoretic motivation without accepting the stronger semantic one outlined. For one can hold that there are inconsistent but non-trivial theories which are interesting, have important applications, useful properties, and so forth, without accepting that they are true. Instrumentalists and formalists would, of course, have no problem in accepting such a theme, though they might *well* find difficulties in clearly distinguishing the stronger, dialethic position from the weaker, more pragmatic, position.[12] Whether either position is tenable on other grounds is another matter, which we will investigate in more detail as we proceed. Indeed, several of the issues raised above will be taken up in more philosophical detail in subsequent introductions. The discussion so far merely serves to indicate some of the motivation for paraconsistency.

2. Approaches to paraconsistent logical theory: Initial systemic taxonomy of paraconsistent logics; zero degree formulas

Having shown that paraconsistent logic is well motivated, we need to specify a paraconsistent logical theory or theories. One thing is perfectly clear, classical two-valued logic is of no use: it is explosive; it is not paraconsistent. Nor for that matter are its extensions such as modal logic. Nor are intuitionist logic or its extensions, for they too are explosive, in virtue of their spread principles such as $A \rightarrow . \sim A \rightarrow B$ and $A \wedge \sim A \rightarrow B$. Many lesser known (but nonetheless significant) logics also fail to meet paraconsistency requirements, and are accordingly logically inadequate to accommodate a range of important philosophical and scientific theories and positions. Among them are various connectional, or broadly relevant logics (i.e. systems satisfying some variable-sharing principle linking antecedents and consequent), in particular connectional logics which validate Disjunctive Syllogism, $A \wedge (\sim A \wedge B) \rightarrow B$, and retain some residual form of Rule Transitivity. Representative of this are conceptivist logics such as Parry systems and connexivist logics.[13] Furthermore, other logics that are technically paraconsistent, such as minimal logic, are not *interestingly paraconsistent* because, although they avoid the disaster of entirely trivializing inconsistent theories, they have the same effect for a whole syntactically determined class of statements. For example, minimal logic would have as holding, in any inconsistent theory at all, all statements of the form $\sim B$ in virtue of its spread principle, $A \rightarrow . \sim A \rightarrow \sim B$.

Beyond this negative data, much less is clear. What should a paraconsistent logical theory be like? There are three fairly well developed answers to this question. What follows will be largely an attempt to explain these three main approaches and to assess, in so far as possible, which is the most viable approach. While no claim is made that these are the only approaches, a well motivated approach fundamentally different from any of these is difficult to envisage, for the following reasons: any adequate logic—adequate that is for the basic relation of deduction—will contain an implication connective, $\rightarrow$, which conforms to *modus ponens*, i.e. A, $A \rightarrow B / B$. Hence any adequate paraconsistent logic will have to break negative paradoxes of implication such as $\sim A \rightarrow . A \rightarrow B$, and there are only so many general strategies for doing this that are compatible with paraconsistency. A first distinction is between approaches that do not break corresponding positive paradoxes, such as $A \rightarrow . B \rightarrow A$ and therefore are characteristically obliged to sacrifice parts of negation theory, in particular Contraposition, and on the other side, approaches that also defeat positive paradoxes. Approaches of the first type, the *positive-plus* approaches, can, like intuitionism, avail themselves of the full strength of Hilbert's positive logic (or extensions thereof), whereas approaches of the latter type, while they can retain negation theory intact, have to adopt a less extravagant positive logic, in effect either some type of modal system or else a relevant positive logic. The modal approach cannot be quite the usual one—though modal substitutivity conditions can be retained, justifying use of the term 'modal'[14]—because paraconsistency requirements would be violated by the following route through conjunction:—

1. $B \wedge \sim B \rightarrow B$, from Simplification, $A \wedge B \rightarrow A$, a modal thesis.

2. $A \wedge {\sim}A \leftrightarrow. B \wedge {\sim}B$, since $A \wedge {\sim}A$ and $B \wedge {\sim}B$ do not differ, e.g. in truth conditions, in any modal (i.e. complete possible) worlds, where $C \leftrightarrow D$ is $(C \rightarrow D) \wedge (D \rightarrow C)$.
3. $A \wedge {\sim}A \rightarrow B$, from 1 and 3 by (modal) substitutivity conditions.
4. $\{C, D\} \models C \wedge D$, i.e. Adjunction, a usual modal rule.
5. $\{A, {\sim}A\} \models B$, from 3 and 4.

Something has to give, and what has given, and had to give, in the modal approach is Adjunction, so yielding the *non-adjunctive* approach to paraconsistency. For to abandon equivalence 2, and accompanying substitutivity, would be to abandon a modal approach, for something in the order of a relevant one, while to reject 1 would be to opt for connexivism, which, since it blocks inference from inconsistency, is not at all congenial to paraconsistency (as we have already noted), and in any case also leads back to a *broadly relevant*, or connectional, approach.

The three main approaches are accordingly, the non-adjunctive approach, the positive-plus approach (of da Costa), and the (broadly) relevant approach. We will investigate these sorts of systems, as far as possible, via their appropriate semantics since these offer, in our view, the clearest understanding of the strengths and weaknesses of the approaches. In so far as paraconsistent logic differs from classical logic, it does so mainly at the propositional level. Hence our discussion will be primarily focussed on the zero order level. Quantifiers and other first order devices can be added in a fairly obvious and straightforward way to all the systems considered. However, at the zero order level it is useful and illuminating to separate the zero degree fragment from the rest. The zero degree fragment of these systems concerns the purely truth functional connectives, $\wedge$, $\vee$, $\sim$, and a number of the important theoretical disagreements between the approaches appear already at this stage. Higher degrees concern the (iterated) behaviour of implication, $\rightarrow$, the issues concerning which are best dealt with separately. Accordingly, we will start our discussion at the zero degree level and reserve the implicational issues until the next section.

2.1. Non-adjunctive systems: Jaśkowski's system

The non-adjunctive approach was pioneered by Jaśkowski[15] (see the Introduction to Part One). The line has been further developed formally by da Costa and a number of co-workers,[16] and has recently appeared again, in thinly disguised form, in the work of Rescher and Brandom[17]. Basically, the idea is as follows: A (piece of) discourse may be produced by a number of different participants. Each contributes to the discourse by producing information which is assumed self-consistent, but which may contradict the information of others. (Perhaps a paradigm example is that of the information presented to a jury at a trial; another example is that of data from different sources fed into a computer.) The things that hold in the discourse (or are true in the discourse) are things which are put forward by some participant.[18] How is this approach to be formalized? We may suppose that each participant has a position. This is the story s/he is prepared to tell, the set of things s/he believes etc. and since this is self-consistent, this can be identified with the set of things true in a classical propositional evaluation, or possible world of standard modal logic. The discourse is just the sum of the participants' positions. Hence the things which hold in the discourse are just the things which hold in any one of the worlds which is a participant's position. Consequently let $\mathcal{M}$ be a possible world model of some modal logic. Let us say S5 for the sake of definiteness. (Different modal logics will give rise to different paraconsistent logics; we will comment where that difference is of any significance.) The definition of 'A holds at world w ($w \models A$)' is as usual. We will define 'A holds discursively in $\mathcal{M}$($\mathcal{M} \models_d A$)' as follows:

$$\mathcal{M} \models_d A \text{ iff for some world w in } \mathcal{M}, w \models A. \qquad (\alpha)$$

We can now define discursive logical validity and discursive logical consequence in the obvious way.

$\models_d A$ iff for all $\mathcal{M}$, $\mathcal{M} \models_d A$.

$\Sigma \models_d A$ iff for all $\mathcal{M}$ either $\exists B \in \Sigma\ \mathcal{M} \not\models_d B$ or $\mathcal{M} \models_d A$.

It is evident that the things which are discursively logically valid are precisely the things which are S5 valid. In particular, a purely truth functional, zero degree formula A is discursively logically valid iff it is a two-valued tautology. By contrast the deducibility relation is anything but classical. For quite clearly $\{A, \sim A\} \not\models_d B$. A countermodel is easy to specify; it simply reflects the picture of discourse with contradictory inputs.

But although the motivation for discursive paraconsistent logic is clear and intelligible, and has good historical roots, there are grave doubts about its adequacy with respect to the basic motivation for paraconsistency. For a start discursive logic fails to be adjunctive. It is easily seen that $\{A \wedge B\} \models_d A$. However, it is equally easy to see that $\{A, B\} \not\models_d A \wedge B$. This means that conjunction has decidedly non-standard behaviour. This by itself may not be a very heavy point. In any paraconsistent logic *something* must behave non-standardly (that is, non-classically). However, in this particular case it casts doubt upon whether conjunction *really is* conjunction in discursive logic. For conjunction just is that connective which has the truth (holding) conditions: $\ulcorner A \wedge B \urcorner$ is true (at a world) iff A is true and B is true (at that world). So something that fails adjunction is not then conjunction. Of course, there is no particular objection to having a non-standard operator '$\wedge$' with curious truth conditions, and hence strange meaning. But it is a serious criticism that '$\wedge$' has *no* recursive truth conditions, i.e. it is impossible to find a condition ψ such that

$$\mathcal{M} \vDash_d A \wedge B \text{ iff } \psi(\mathcal{M} \vDash_d A, \mathcal{M} \vDash_d B).$$ [19]

A more important point is, however, that there can be no objection to there being a genuine conjunction in the system. Perhaps, then, discursive logic just suffers from an omission? Suppose we add a genuine conjunction to the language with the semantic conditions

$$\mathcal{M} \vDash_d A + B \text{ iff } \mathcal{M} \vDash_d A \text{ and } \mathcal{M} \vDash_d B. \qquad (\beta)$$

The problem now is how to define the truth conditions of truth functions of sentences of the form A+B. There are two possibilities.

The first is that we can find some formula of the unaugmented modal language with two propositional parameters, with which + can be identified. Condition (α) then provides the truth conditions of formulas in a straightforward way. This approach has been adopted by da Costa, who defines discursive conjunction, using the possibility functor M, thus:

$$A \wedge_d B = MA \wedge B.$$ [20]

Discursive conjunction can quickly be seen to satisfy condition (β), at least in S5. Actually the lack of symmetry between A and B in the definition of $\wedge_d$ is displeasing and makes the definition appear to float in mid-air. It would be clearer to define

$$A + B = MA \wedge MB. \qquad (\gamma)$$

Condition (β) is still satisfied.

The problems with this approach to conjunction are two-fold. First, it is totally opaque why a modal functor such as M should poke its nose into the meaning of ordinary extensional conjunction. Granted that (β) fixes the extension of +, (γ) fixes the sense. This makes it quite clear that + is not ordinary conjunction, even though it has the right extension.

Secondly, this approach to conjunction totally destroys the normal relationships between conjunction, disjunction and negation. For example, none of the following holds:

$$\{\sim A\} \vDash_d \sim (A+B); \ \{\sim A + \sim B\} \vDash_d \sim (A \vee B); \ \{\sim A \vee \sim B\} \vDash_d \sim (A+B).$$

This is little more than the consequence of the fact that a modal functor has got embroiled in conjunction. It is worth saying again that some classical logical relations will have to go paraconsistently. However, the wholesale destruction of the relations normally taken to hold between conjunction, negation and disjunction clearly speaks against discursive conjunction. This is especially true when there are other options (such as the relevant one) which preserve virtually all these relations.

The other possibility is to refuse to identify A+B with any sentence functor of the unaugmented language, but to give the truth conditions of compounds of + sentences in the usual way, e.g.

$$\mathscr{M} \vDash_d \sim(A+B) \text{ iff } \mathscr{M} \nvDash_d A+B$$

and similarly for conjunction and disjunction. This, at least, preserves all the classical relations between conjunction, disjunction and negation. However, it runs into other problems. In particular, it reinstates a very general form of non-paraconsistency. For it is now easy to see that

$$\{A+B, \sim(A+B)\} \vDash_d C$$

and as a special case

$$\{A+A, \sim(A+A)\} \vDash_d C.$$

Now, not only is it difficult to discern a connection between the premisses and the conclusion, but this is little better than the full, horrible, *ex falso quodlibet.* Should one participant in a discourse say 'It is raining and it is raining' and *another* say, 'No, that's not the case', the whole thing, quite counterintuitively, blows up. No one who takes paraconsistency *seriously* can accept this option.

The second objection to approaching paraconsistent logic discursively is more damaging than the first. It concerns the relation of logical consequence which is (as befits a paraconsistent logic!) both too strong and too weak.

First it is too strong. It is easily seen that $\{A\} \vDash_d B$ iff B is a classical two-valued consequence of A. This means that discursive logic is only half-heartedly paraconsistent. For everything does follow discursively from a conjoined contradiction: $\{A \wedge \sim A\} \vDash_d B$. What stops discursive logic from lapsing into non-paraconsistency is just the non-standard behaviour of conjunction. Because single premiss discursive validity coincides with classical validity, discursive logic is extremely badly suited to be the underlying logic of some of the most important inconsistent theories.[21] For example, classically $\{\exists y \forall x(x \in y \leftrightarrow x \notin x)\} \vDash \exists y(y \in y \wedge y \notin y)$, and $\exists y(y \in y \wedge y \notin y) \vDash B$. Hence if Σ is the set of instances of the abstraction scheme of set theory, $\Sigma \vDash_d B$. Thus discursive paraconsistent logic is totally unsuitable as the underlying logic of naive set theory. Similarly it is unsuitable as the underlying logic of naive semantics.

Rescher and Brandom try to avoid this difficulty[22] by suggesting that instances of the abstraction scheme which give rise to trouble be split into two halves. Thus the instance generating the Russell paradox becomes the pair comprising $\forall x(x \in R \rightarrow x \notin x)$ and $\forall x(x \notin x \rightarrow x \in R)$. Set theory is then split into essentially two distinct theories, one of which contains the first of these and the other of which contains the second. Each of these two theories then holds in a different possible world.

In fact this strategy has only the appearance of paraconsistency. In essence it is just a revisionist classical position. For paring an inconsistent theory down to various consistent subtheories is a game[23] that classical set theorists have been playing for eighty years. The classicist is quite happy with both the above fragments of set theory. Hence this line does not take the first motivation, for inconsistent theories (as opposed to consistent fragments of inconsistent theories), seriously. All that Rescher and Brandom add to the classical position is the insistence that both fragments be true. However, the classicist will understand this as 'true in some possible world' and there will be no disagreement. Neither can the discursivist really object to the classicist understanding. For this is, in effect, what his understanding of paraconsistent truth amounts to as well.

The other side of this objection to discursive logical consequence is that it is too weak. To be exact, let Σ be a non-null set of zero degree formulas and let A be a first degree formula. Then, if $\Sigma \models_d A$ there is some $B \in \Sigma$ such that $\{B\} \models_d A$. To see this, suppose for *reductio* that there is no $B \in \Sigma$ such that $\{B\} \models_d A$. Then for every B we can find a model $\mathcal{M}_B$ such that, for some world w in $\mathcal{M}_B$, B is true in w, whilst for no world w, A is true in w. Let $\mathcal{M}$ be the collection of all the worlds in every $\mathcal{M}_B$. Then $\mathcal{M}$ is countermodel to $\Sigma \models_d A$.

Hence there is no such thing as a valid multi-premiss discursive inference![24] This shows that as a logic for drawing inferences in real life situations, discursive logic is useless. (This too is important since one of the main motivations for paraconsistency was that useful conclusions should be drawn from actual inconsistent data, e.g. laws, judicial evidence, etc. Paraconsistent logic should, as Jaśkowski puts it, 'be rich enough to enable practical inference'.)[25] For no premisses can be combined to draw conclusions. Conceivably we might consider each of the participants in a discourse to be offering one long conjoined statement. However, by the very motivation, the contributions of each participant are not to be considered as conjoined. What follows in a discourse is all and only what follows from the contribution of any one participant. (The judge cannot infer from the statements of witness A that Jones was in the room and of witness B that no one else was in the room that Jones was the only person in the room!) This shows that discursive logic is not really acceptable even according to its own *rationale*, namely the drawing of reasonable inferences from inconsistent data from different sources. In fact both the other approaches to paraconsistency we will consider are better suited to this end.

We can sum up the foregoing discussion simply. Discursive logic may be either single premiss or multiple premiss. In the first case it is classical. In the second it is really no logic at all. In neither case it is suitable for the investigation of inconsistent theories. The main problem with the discursive approach is just that it does not take the second, dialethic, motivation (that there are true contradictions) seriously. Contradictions may be "true" but this amounts to no more than "true in different worlds". Moreover each possible world is as consistent as any classicist could wish: the approach is much too modally based to accommodate inconsistency satisfactorily.[26] For all these sorts of reasons, the non-adjunctive modal approach to paraconsistency should be dismissed.

2.2. Positive-plus systems: da Costa's main systems

The most detailed study of positive-plus system was initiated (as we saw in a previous introduction) by da Costa, who proposed a family of paraconsistent logics C_i, where $1 \leq i \leq \omega$.[27] The systems differ in points of detail but share the same basic semantical motivation. In fact the axiom systems came first and the semantics only later.[28] But the semantics are the most illuminating path to da Costa's approach, so we will concentrate on these, and in particular the semantics of system C_ω.

Unlike discursive logic, da Costa does take the idea that there are true contradictions seriously. Da Costa formalizes this as follows. Given a propositional language, a *da Costa evaluation* is a function ν which maps every formula to 1 (true) or 0 (false) satisfying the conditions

(1) $\nu(A \wedge B) = 1$ iff $\nu(A) = 1$ and $\nu(B) = 1$
(2) $\nu(A \vee B) = 1$ iff $\nu(A) = 1$ or $\nu(B) = 1$
(3) $\nu({\sim}A) = 1$ if $\nu(A) = 0$
(4) $\nu(A) = 1$ if $\nu({\sim}{\sim}A) = 1$

There are also conditions for $\supset$ too. We will consider these later. The above conditions can be shown to be characteristic for the zero degree part of C_ω. The conditions for $\wedge$, $\vee$ are normal ones and ensure that these really are conjunction and disjunction. The deviation from classical logic is only in the conditions for $\sim$. (3) ensures that at least one of A, ${\sim}$A is true (though both may be). And the rationale for (4) seems to be something like this: if it is not the case that not-A then since (by (3)) one of A and ${\sim}$A is true, A is true. Logical truth and consequence are defined in the usual way:

$\Sigma \models_C A$ iff for all evaluations ν, either $\nu(A) = 1$ or for some $B \in \Sigma$ $\nu(B) \neq 1$.
$\models_C A$ iff for all evaluations ν, $\nu(A) = 1$.

Quite clearly, neither $\{A, \sim A\} \models_C B$ nor $\{A \wedge \sim A\} \models_C B$.

The problems with da Costa's approach are perhaps not so obvious as those with the non-adjunctive systems. However, in the end they are equally telling.

The first objection is that condition (4) of the da Costa semantics is ill-motivated. (4) appears to follow from (3). (It does not, since otherwise it would be redundant.) The argument gets by by reading '$\nu(\sim\sim A) = 1$' as 'It is not the case that $\sim A$' and then supposing that the latter means $\nu(\sim A) = 0$. This is a fallacy of equivocation since the inference from $\nu(\sim\sim A) = 1$ to $\nu(\sim A) = 0$ is invalid even in da Costa's terms.

Without this argument, the motivation for condition (4) is totally obscure on this approach. If the truth values of A, $\sim A$, and $\sim\sim A$ are independent enough to let all be true, why shouldn't they be independent enough to let the first be false and the last two be true? Compare this with the next approach we deal with, where the connection in truth-value between a sentence and its negation falls, quite naturally, out of the motivating considerations. Of course condition (4) could be dropped from da Costa's semantics. However in that case, negation would have virtually none of the properties traditionally associated with negation. (It has few enough anyway.) This would strengthen our subsequent argument that da Costa's negation is not really negation at all.

The second objection to da Costa semantics is that they are non-recursive. Now whilst non-recursive semantics may be admirable for many technical purposes, there are good reasons for not being philosophically satisfied with them. The arguments are well known, but the crucial point is something like this: since speakers of a language are able to understand sentences they have never heard before, the sense or meaning of a sentence must be determined by the senses of its components. In particular, then, an adequate semantics must specify recursively the meaning of a sentence in terms of the meanings of its components. Thus generally speaking the specification of semantic conditions must be recursive. Now da Costa semantics are certainly not recursive since the truth conditions of $\sim A$ are not determined by the truth conditions of A. (If $\nu(A) = 1$, $\nu(\sim A)$ could be 1 or 0.) Thus these semantics have problems. This argument against da Costa semantics is not completely conclusive. It could be met by arguing that meaning is not completely determined by truth conditions, and that some other factor, let us call it *sense*, is involved. It can then be argued that whilst meaning conditions are recursive the truth conditions of a compound may depend upon the sense (rather than the truth value) of its components. In particular the truth value of $\sim A$ may be determined by the "sense factor-X" of A.

This general approach to meaning is of course the one adopted in Montague semantics. We will not detour to examine the adequacy of this general approach to the theory of meaning. For it is enough to observe the following: First, even if this approach could be made to work (and it cannot in general[29]), da Costa semantics, as they stand, are radically incomplete. Secondly, if this approach were to work it would show that $\sim$ is not our friendly neighbourhood extensional negation, but a radically intensional functor of some sort. Of course this point may be countered, but it is the first bit of evidence we will muster to show that da Costa negation is not really negation.

Let us turn from da Costa valuations to the set of zero degree logical truths in da Costa's approach. Since every classical evaluation is a da Costa evaluation then we have that if $\vDash_C A$, A is a classical two-valued tautology. The converse however, is not true. The most notable exception is the law of non-contradiction:

$$\sim(A \wedge \sim A) \qquad (\varepsilon)$$

The omission of this from a system of paraconsistent logic is not surprising. Nor is it a coincidence that it happens in da Costa's system; for he lays down as a condition of adequacy on a paraconsistent logic that (ε) not be valid.[30] The rationale for the omission of (ε) appears to be clear enough: some statements of the form $A \wedge \sim A$ *are* true. However, we should proceed with care. This does not settle the matter—even by da Costa's standards. For the fact that $A \wedge \sim A$ is true does not prevent $\sim(A \wedge \sim A)$ from being true too. In fact, insisting that the absence of (ε) be a condition of adequacy on a paraconsistent logic is far too strong. It is quite open for a paraconsistentist to adopt (ε), as the next approach we examine will show. Of course if we do adhere to (ε) then any contradiction $A \wedge \sim A$ (let us call this a *primary contradiction*) will generate another $(A \wedge \sim A) \wedge \sim(A \wedge \sim A)$ (let us call this a *secondary contradiction*). However, obviously there is no *a priori* bar to this for the paraconsistentist.

Is it best then to hold on to (ε) or to reject it? We do not wish to be too dogmatic about this. However, presumably any case against (ε) will hinge on the undesirability of secondary contradictions. Conceivably we might invoke the razor that contradictions should not be multiplied beyond necessity. However, even if this is correct (and is it?) it does not get us very far until we know what "necessity" is. We think the case in favour of (ε) much more plausible. Part of it goes like this. The law of non-contradiction has traditionally been seen as a central property, if not a defining characteristic, of negation. And this is true not only of traditional and classically oriented logicians such as Aristotle and Russell, but also of those who believed in true contradictions such as Hegel.[31] That an account of negation violates the law of non-contradiction therefore provides prima facie evidence that the account is wrong. This is the second piece of evidence that *da Costa negation is not negation.*

In fact, we can make the claim more precise. Traditionally A and B are sub-contraries if $A \vee B$ is a logical truth. A and B are contradictories if $A \vee B$ is a logical truth *and* $A \wedge B$ is logically false. It is the second condition which therefore distinguishes contradictories from sub-contraries. Now in da Costa's approach we have that $A \vee \sim A$ is a logical truth. But $A \wedge \sim A$ is not logically false. Thus A and $\sim A$ are sub-contraries, not contradictories. Consequently da Costa negation is not negation, since negation is a contradiction forming functor, not a sub-contrary forming functor.

Let us now turn our attention to the relation of logical consequence. Again it is easily seen that this is a sub-relation of classical two-valued logical consequence. However, the following *fail*, showing that it is a proper sub-relation.

$$\begin{array}{ll} \{\sim A\} \vDash_C \sim (A \wedge B) & \{\sim (A \vee B)\} \vDash_C \sim A \\ \{A\} \vDash_C \sim\sim A & \{\sim A \wedge \sim B\} \vDash_C \sim (A \vee B) \\ \{\sim A \vee \sim B\} \vDash_C \sim (A \wedge B) & \end{array} \qquad (\kappa)$$

Moreover as we shall be able to see later the following also fail.

$$\begin{array}{l} \{A \supset B\} \vDash_C (\sim B \supset \sim A) \\ \{A \supset B, A \supset \sim B\} \vDash_C \sim A \end{array} \qquad (\lambda)$$

This shows that da Costa negation has virtually none of the inferential properties traditionally associated with negation. (Compare this with negation in the next approach we consider, which has all the above properties.) This is a further piece of evidence suggesting that da Costa negation is not really negation. We have now mustered strong evidence to this effect and the case seems pretty conclusive. It is time to ask what da Costa negation *is.*

The key to this problem is provided by our discussion of the logical truths. We saw there that da Costa negation behaves like a sub-contrary forming operator, not a contradictory forming operator. Indeed, the truth conditions of negation (3) make this reading of $\sim$ almost mandatory. Hence we suggest that da Costa's negation is an operator which turns a formula into a sub-contrary. This not only explains the truth condition of $\sim$ and the behaviour of logical truths, but is also well confirmed for other reasons. First, if $\sim$ is a sub-contrary forming operator then we should expect all the inferential principles (κ), (λ) to fail, which they do. Secondly, this fact explains why $\sim$ is not truth functional. For the truth value of a subcontrary of A is not determined by the truth value of A. Thus $\sim$ is not an extensional functor. All this fits the picture.

So, $\sim$A is a sub-contrary of A, but which? For although the contradictory of a statement is unique, it may have many sub-contraries. Which is $\sim$A? It must be a sub-contrary which satisfies condition (4) of the semantics. However, this is by no means sufficient to determine the functor $\sim$ uniquely. If A and B are *any* sub-contraries then the functor which maps A to B and vice versa satisfies this condition. There are no other constraints on $\sim$ to determine which sub-contrary functor it is. Hence the answer to this question must be radically indeterminate.

Is the lack of a genuine negation operator in the C systems merely a matter of omission? The answer is a quick and simple 'No'. For if we were to add an operator, $-$, with the obvious conditions for negation,

$$\nu(-\mathrm{A}) = 1 \text{ iff } \nu(\mathrm{A}) = 0,$$

it is easy to see that non-paraconsistency would be reinstated. For then $\{\mathrm{A} \wedge -\mathrm{A}\} \vDash_{\mathrm{C}} \mathrm{B}$. Thus the C systems achieve their paraconsistency only at the cost of dispensing with negation.

So much for da Costa's general approach to zero degree formulas—points that rub off on to the more comprehensive positive-plus approach. Before we set such approaches aside, however, it is worth discussing the way da Costa strengthens system C_ω to produce the systems C_i, $1 \leq i < \omega$. For the sake of definiteness we will fix our attention on C_1 (though all the points made apply equally to the others).

It is clear that on a da Costa evaluation there are two kinds of statements: those that are "paradoxical", i.e. those such that $\nu(\mathrm{A}) = \nu(\sim\mathrm{A}) = 1$ and those that are "classical", i.e. such that $\nu(\mathrm{A}) \neq \nu(\sim\mathrm{A})$. Although classical logic does not hold for all sentences, it would be reasonable to suppose that it holds for sentences with classical values. (Actually in C_ω it does not.) Moreover, it is reasonable enough to suppose that this should in some sense be expressible in the language itself. In particular suppose we write 'A^0' for 'A has a classical truth value', then the following is reasonable:

If B is a compound of $\mathrm{A}_1 \ldots \mathrm{A}_n$ and $\Gamma \vDash_{\mathrm{C}} \mathrm{A}_1^0 \wedge \ldots \wedge \mathrm{A}_n^0$ then
$\Gamma \vDash_{\mathrm{C}} \mathrm{B}$ iff B is a classical consequence of Γ. (ξ)

Achieving this is precisely the motivational move from C_ω to C_1.

First, a "classicality" operator ^{0} has to be produced. There are two different approaches possible here. The syntactic approach is to identify 'A is paradoxical' with '$\mathrm{A} \wedge \sim\mathrm{A}$' and hence define 'A has a classical value (A^0)' as '$\sim(\mathrm{A} \wedge \sim\mathrm{A})$'. The semantic approach is to give the truth conditions of A^0 directly as:

If $\nu(A) \neq \nu(\sim\mathrm{A})$, $\nu(\mathrm{A}^0) = 1$
If $\nu(\mathrm{A}) = \nu(\sim\mathrm{A}) = 1$, $\nu(\mathrm{A}^0) = 0$.

Da Costa wants to follow both these approaches. He defines A^0 as $\sim(A \wedge \sim A)$. From this, the first of the semantic conditions follows. For if $\nu(A) \neq \nu(\sim A)$, $\nu(A \wedge \sim A) = 0$ and $\nu(A^0) = \nu(\sim(A \wedge \sim A)) = 1$. However, the second does not follow in C_ω semantics. Hence it has to be enforced with a new semantical postulate to this effect:

$$\text{If } \nu(A \wedge \sim A) = 1,\ \nu(\sim(A \wedge \sim A)) = 0.^{32} \qquad (\mu)$$

The only other semantical postulates for C_1 ensure that all formulae compounded entirely from formulae with classical values have classical value, thus:

$$\text{If } \nu(A^0) = \nu(B^0) = 1 \text{ then } \nu((A \wedge B)^0) = \nu((A \vee B)^0) = \nu((A \supset B)^0) = \nu((\sim A)^0) = 1.$$

These conditions ensure that (ξ) holds,[33] thus fulfilling the motivation.

There are a few points to make about extending C_ω with a "classicality" operator. The first is that it in no way affects our conclusions about the interpretation of da Costa negation. Even in C_1, negation is non-extensional, the law of non-contradiction still fails and so do all the principles of inference (κ) and (λ). Thus our conclusion that $\sim$ is a sub-contrary forming operator still stands. (Although, of course, the extra semantic constraints on $\sim$ add some further constraints on which sub-contrary of A, $\sim$A can be.) The second and more important point is that the addition of a classicality operator in this way leads to new trouble. It is easily checked that for any formula B, $\nu(B \wedge \sim B \wedge B^0) = 0$ for any ν. Thus

$$\{B \wedge \sim B \wedge B^0\} \vDash_{C_1} A. \qquad (\zeta)$$

Hence if we can ever prove a theorem of the form $B \wedge \sim B \wedge B^0$, things reduce to triviality. But it is easy to produce a theorem of this form in a semantically closed language (as da Costa has noted).[34] By the usual self-referential construction we can find a statement β such that $\beta \leftrightarrow (\sim\beta \wedge \beta^0)$. (This sentence is false and has a classical truth value.) It is then easy to prove with reasoning valid in C_1 (in fact in C_ω) $\beta \wedge \sim\beta \wedge \beta^0$. A similar argument can be performed in naive set theory. Hence C_1 is entirely unsuited to formalizing two of the most important paraconsistent theories. Moreover, the trouble extends much more widely, to other paraconsistent theories, and to restricted forms of set theory.[35]

This is, of course, a major additional argument against the C_1 approach to paraconsistency. However, there is a more general lesson to be learnt here. Since the situation does not arise in the same way in C_ω, the problem lies with the classicality operator. We can locate the trouble more precisely. In the proof of $\beta \wedge \sim\beta \wedge \beta^0$, the only fact specifically about β^0 that is used is that $\sim(\beta^0) \rightarrow \beta \wedge \sim\beta$ and this is guaranteed by the syntactic definition of A^0. What produces the special case of *ex falso quodlibet* (ζ) is precisely the semantic conditions of B^0. Thus we see that the semantic approach to a classicality operator and the syntactic approach are incompatible. Plausible as both may seem, one has to give. Let us move on to the third approach to paraconsistency. This is the relevant one taken by both authors.[36]

2.3. *The relevant approach*

Semantically there are several ways of proceeding. We will choose one that has seemed (especially to the more classically-inclined) particularly simple to grasp.[37] Like da Costa's approach, the relevant approach takes seriously the view that some statements are true *and* false. However, instead of insisting that every sentence take a unique truth value, it allows statements to have both.

Formally, let $V = \{\{1\}\{0\}\{1, 0\}\}$. Here $\{1\}$ is (the classical) true and true only; $\{0\}$ is (the classical) false and false only; $\{1, 0\}$ is (the paradoxical) true and false;

A valuation is a map ν from the set of zero degree formulas to V such that

1a) $1 \in \nu(\sim A)$ iff $0 \in \nu(A)$ | b) $0 \in \nu(\sim A)$ iff $1 \in \nu(A)$
2a) $1 \in \nu(A \wedge B)$ iff $1 \in \nu(A)$ and $1 \in \nu(B)$ | b) $0 \in \nu(A \wedge B)$ iff $0 \in \nu(A)$ or $0 \in \nu(B)$
3a) $1 \in \nu(A \vee B)$ iff $1 \in \nu(A)$ or $1 \in \nu(B)$ | b) $0 \in \nu(A \vee B)$ iff $0 \in \nu(A)$ and $0 \in \nu(B)$

Logical truth and consequence are defined in the obvious way.

$\Sigma \models_R A$ iff for all evaluations ν either $1 \in \nu(A)$ or for some $B \in \Sigma$, $1 \notin \nu(B)$
$\models_R A$ iff for all evaluations ν, $1 \in \nu(A)$.

It is easy to see that these truth conditions are paraconsistent, i.e. that $\{A, \sim A\} \not\models_R B$. Moreover, the truth conditions look very familiar. Indeed they are just the classical ones. Of course in the classical case the second one of each pair is redundant. However, this is no longer the case when we have grasped the paraconsistent insight that things may be both true and false.

Some of the more important features of the deducibility relation are as follows:

$\{A, B\} \models_R A \wedge B$ | $\{A \wedge B\} \models_R A$

$\{A\} \models_R A \vee B$ | If $\{A\} \models_R C$ and $\{B\} \models_R C$ then $\{A \vee B\} \models_R C$

If $\{A\} \vDash_R B$ and $\{B\} \vDash_R C$ then $\{A\} \vDash_R C$

$\{\sim A\} \vDash_R \sim(A \wedge B)$

$\{\sim A, \sim B\} \vDash_R \sim(A \vee B)$

$\{\sim(A \vee B)\} \vDash_R \sim A$

$\{A\} \vDash_R \sim\sim A$

$\{\sim\sim A\} \vDash_R A$

$\{\sim A \vee \sim B\} \vDash_R \sim(A \wedge B)$ (and all the other De Morgan principles).

Moreover it is straightforward to establish that $\vDash_R A$ iff A is a two-valued classical tautology.[38]

These properties make it easy to see that this approach avoids the problems of the two previous approaches. Unlike the non-adjunctive systems, it has an adequate conjunction and a decidedly non-trivial multi-premise deducibility relation. The properties of negation are neat and simple and no extra semantic postulates have to be added, as in da Costa's approach, to ensure bits of double negation. Moreover, there can be no doubt that the negation of this approach *is* negation. The semantics are recursive and extensional. Thus $\sim$ is not an intensional functor. Both the laws of excluded middle and non-contradiction hold and negation has all the deducibility relations one would expect.[39] Someone might try to make out that the negation of this system is not really negation. But in virtue of all the above points, they would have little ground to stand on. The negation of A is that statement which is true if A is false and false if A is true. But this is exactly what the relevant truth conditions say.

A pleasing feature of the semantics is that the set of zero degree logical truths is exactly the set of classical tautologies. This shows that this is a particularly stable set of formulas valid in both classical and inconsistent contexts. Moreover, it shows that in a sense relevant paraconsistent logic subsumes classical logic at its zero degree level.

Turning to the deducibility relation, it is easy to see that this is a sub-relation of the classical one. Indeed on pain of non-paraconsistency, this must be a proper sub-relation. Those running through the list of valid consequences given above, and not familiar with relevant logic, might wonder exactly what of classical logic is paraconsistently invalid. The answer is that it is the principle of the disjunctive syllogism

$\{A, \sim A \vee B\} \vDash B$

and its cognates such as

$\{A, \sim(A \wedge B)\} \vDash \sim B.$

This is in fact the only major principle of classical inference that is rejected on the relevant paraconsistent approach. Despite this, its rejection has drawn some fire from various sources. A full discussion of the issue would involve a considerable detour.[40] However, a few points are worth making. First, as we pointed out right at the beginning, if paraconsistency is to be taken seriously, something of classical logic has to be rejected. It is therefore no argument against this approach *per se* to point out that the disjunctive syllogism is rejected. Indeed the relevant approach holds the losses from classical logic to a minimum at the zero degree level. Both of the other approaches we have considered lose the disjunctive syllogism and much else besides. This is the only loss on the relevant paraconsistent position. Moreover, the loss of the disjunctive syllogism is not as great a blow as might be thought. First, many of the cases of disjunctive syllogism occurring in natural practice use an intensional 'or', $\veebar$. This can be defined simply

$$A \veebar B = {\sim}A \rightarrow B$$

The intensional disjunctive syllogism

$$\{A, {\sim}A \veebar B\} \models B$$

is certainly valid.[41] In fact it is little more than *modus ponens.*

The second and more important reason is that although the disjunctive syllogism is generally invalid, it is usable in certain contexts. The point needs to be handled with some care as a later paper in this collection[42] shows. However, basically the point is this. The reason that the disjunctive syllogism fails is that the sentence A may be paradoxical. If A and ${\sim}A$ are true, then so are A and ${\sim}A \vee B$, whatever B is. However, if this case is ruled out no more counterexamples to the disjunctive syllogism can be produced. Thus, provided we are not in a paradoxical situation (i.e. one where A is both true and false), the disjunctive syllogism can legitimately be used. Now it is easy to see that if the disjunctive syllogism is added to zero degree relevant paraconsistent logic, classical logic results. Hence what we see is that in non-paradoxical, consistent contexts (which are of course the only ones countenanced by classical logic anyway) classical logic is acceptable. Thus the general failure of the disjunctive syllogism is not a serious problem.

With the rejection of this—perhaps the major objection to relevant paraconsistent logic—we conclude that the relevant approach is the best one to paraconsistency, at least at the zero degree level.

One final point: if one admits truth-value gluts (i.e. statements that are both true and false), it might seem natural to accept truth-value gaps (i.e. statements that are neither). In fact all the approaches to paraconsistency we have discussed can be modified in fairly obvious ways to allow for this possibility. However, the matter of truth-value gaps is a separate issue, in no way entailed by the paraconsistent position. Accordingly, the issues raised by the modification of these logics to allow for truth-value gaps are not, strictly speaking, relevant to paraconsistency. It is for this reason that we can avoid opening this problem here.[43]

3. Approaches to paraconsistent logical theory: implication

So far we have concentrated on features of the various approaches to paraconsistency at the zero degree level. However, all the approaches have distinctive implication operators. This is no accident. Implication is a central logical connective. Any adequate logic must give an account of its behaviour. The classical analysis of the implication operator $\rightarrow$ identifies $A \rightarrow B$ with $A \supset B$ (i.e. $\sim A \vee B$). This results in an equation of *modus ponens*, $A, A \rightarrow B/B$, with the disjunctive syllogism, which (as we saw at the end of the last section) fails in all the semantical approaches we considered. Yet *modus ponens* is the fundamental principle governing implication. No operator which fails to satisfy this can be implication. Hence each of the approaches must find a different, non-classical, account of implication.

3.1. Non-adjunctive systems, such as Jaśkowski's system

Since the non-adjunctive approaches use possible world semantics, the natural implication operator in this context would certainly seem to be strict implication. Let us define $A \rightarrow B$, as usual, to be $\Box(A \supset B)$. Then it is easily verified that $\{A, A \rightarrow B\} \vDash_d B$. Observe that although strict implication suffers from paradoxes which *appear* to make it unsuitable for paraconsistency (e.g. $(A \wedge \sim A \rightarrow B)$), this is not the case *given* that adjunction fails. Jaśkowski is well aware of this possible definition of implication,[44] though he opts for another possibility which we will discuss shortly. Nonetheless, we should ask whether strict implication is a satisfactory implication operator in the context of discursive logic.

The answer is that it is not. The first point is that although *modus ponens* holds for strict implication if the underlying modal logic is S5, it fails for weaker logics. However, the two most important objections are ones which we will meet several times in this part; hence it is worth giving them names.

The first objection is the *irrelevancy objection.* The point here is that an implication should hold between A and B only in virtue of some common content between A and B. The truth value of an implication should not depend simply upon the truth value of one of its components, nor on the modal value of one of its components. Implication is *essentially relational.*

This, though fairly banal, runs against classical (though not traditional) orthodoxy. It is a mark of the extent to which indoctrination of the classical view has been effective, that the irrelevancy of classical logic has not been seen as a defect in need of a remedy, and that vast amounts of argument[45] have been necessary to try to reopen people's eyes to the point and to reorient vision towards the True. However, given the enormous amount that has been written on relevance, it would be otiose for us to argue the case for it here again. Let us therefore merely endorse, or re-endorse, the arguments of Anderson, Belnap, Meyer, ourselves and many others, that implication is relevant. Now relevant logic and paraconsistent logic are not the same thing. It is possible to have irrelevant paraconsistent logics (as we are just about to see) and vice versa.[46] Hence relevance is not *de rigeur* for a paraconsistentist. However, while we are in the process of reworking logic we might as well get implication right—in which case irrelevance is a failure of a paraconsistent logic.

So far so good. But what exactly *is* the relevance requirement? Again this is a deep question and, since this is a book about paraconsistency, one that we can fortunately largely avoid. For present purposes all that is necessary is a test for irrelevance, and for this the Anderson and Belnap variable-sharing test will do nicely.[47] A sufficient condition for a (purely) propositional logic to be irrelevant is that it have a theorem (logical truth) of the form $A \to B$ where A and B have no propositional variable in common. In such a case A and B have no common content. Having got this far it is now easily seen that strict implication is irrelevant, even in a discursive context. For $\models_d (A \wedge \sim A) \dashv B$, $\models_d B \dashv (A \vee \sim A)$, and all the other horrors of strict implication. Thus this approach to implication fails the relevancy objection.

The second objection is the *Curry objection.* There is an argument, due to Curry,[48] which shows that under certain conditions, naive set theory and semantics are trivial, that is, anything can be proved in them. The argument can be put in a number of different forms. Here is one of them.

Let β be the sentence 'If this sentence is true, A is' where A is arbitrary, i.e.

$$\beta = \ulcorner \mathrm{Tr}\beta \to A \urcorner.$$

By the truth scheme of naive semantics

$$\mathrm{Tr}\beta \leftrightarrow (\mathrm{Tr}\beta \to A). \quad (1)$$

Hence by absorption $(C \to (C \to D))/C \to D$ from left to right

$$\mathrm{Tr}\beta \to A. \quad (2)$$

So by (1), (2) and *modus ponens*

$$\mathrm{Tr}\beta \tag{3}$$

and by (2), (3) and *modus ponens*

A.

Thus if naive semantics is based on a logic which contains *modus ponens* and absorption, it is trivial. A similar result holds for naive set theory. Now one of the main motives for paraconsistent logic was the investigation of interesting inconsistent theories, of which naive set theory and semantics are perhaps the two most interesting. Thus any logic which contains both *modus ponens* and absorption is an unsuitable paraconsistent logic. In fact, since *modus ponens* is essential to any implication operator, it follows that a paraconsistent logic is objectionable if it contains absorption.

It is easily seen that absorption is true of strict implication, i.e. $\{A \rightarrow\!\!\!\!\!\rightarrow (A \rightarrow\!\!\!\!\!\rightarrow B)\} \vDash_d A \rightarrow\!\!\!\!\!\rightarrow B$. Hence this is not a suitable paraconsistent implication.

The third and final objection we will present against strict implication is similar to the second but a bit more parochial. For an additional reason, strict implication is quite unsuited for the role of the underlying implication of naive set theory and semantics. This is because $\{A \leftrightarrow\!\!\!\!\!\rightarrow \sim A\} \vDash_d B$, where $\leftrightarrow\!\!\!\!\!\rightarrow$ represents strict coimplication. An application of the abstraction axiom of naive set theory (or the truth scheme of naive semantics), with the implication operator being considered as strict implication, yields $\{x|x \notin x\} \in \{x|x \notin x\} \leftrightarrow\!\!\!\!\!\rightarrow \sim \{x|x \notin x\} \notin \{x|x \notin x\}$, whence, again naive set theory and semantics are trivial.

As we said before, Jaśkowski did not accept the obvious modal candidate, strict implication, as an account of implication. His candidate for this, called 'discursive implication' ($\supset_d$) is defined as follows:

$$A \supset_d B \text{ iff } MA \supset B.$$

If we recall that the things true at some possible world are the story or position of some participant in the discourse, we can understand Jaśkowski's gloss of $A \supset_d B$ as 'if anyone states that A, then B'.[49] Leaving aside the question of the adequacy of this gloss, it is easy to check that discursive implication at least satisfies *modus ponens*:

$$\{A, A \supset_d B\} \vDash_d B.$$

However, discursive implication fares little better than strict implication. It is straightforward to establish, as Jaśkowski himself did,[50] the following fact:

Let A be any formula which contains only the connective $\supset$, and let A_d be A, with every occurrence of '$\supset$' replaced by '$\supset_d$'. Then $\models_d A_d$ iff A is a two-valued tautology. The proof is as follows. Suppose that A is a two-valued tautology. We need to show that for all $\mathcal{M}$, A_d holds in $\mathcal{M}$, which, by the completeness theorem for S5 is true iff MA_d is a theorem of S5. Now consider MA_d. It is easily checked that $\models_{S5} M(A \supset_d B) \leftrightarrow (MA \supset MB)$. By repeated application of this strict equivalence we can drive all the 'M''s in MA_d inwards as far as possible, replacing all '$\supset_d$''s with '$\supset$''s. We then end up with a formula which is a substitution instance of A, which is certainly provable in S5. Conversely, suppose that A is not a tautology. Let w be a classical world at which it fails and let $\mathcal{M}$ be the model which contains only that world. Then $\mathcal{M} \models MB \leftrightarrow B$ and since A_d is obtained from A by the suitable insertion of M's, $\mathcal{M} \not\models A_d$.

Thus the pure calculus of discursive implication is just the pure calculus of material implication. It is not true that the $\sim$, $\wedge$, $\vee$, $\supset_d$ fragment of discursive logic is identical with classical logic. (For example, it is easily checked that $\not\models_d A \supset_d (\sim A \supset_d B)$.) However, the initial result is damaging enough. For it shows, first, that discursive implication, like material implication, falls to the irrelevancy objection since for example $\models_d A \supset_d (B \supset_d B)$, and second, that discursive implication falls to the Curry objection since $(A \supset (A \supset B)) \supset (A \supset B)$ is a classical tautology.

Discursive implication does not fall to other objection mooted against strict implication, but only because of a sleight of hand on Jaśkowski's part. Suppose we were to define discursive equivalence $\equiv_d$ in the obvious way, viz. $A \equiv_d B = (A \supset_d B) \wedge (B \supset_d A)$; then it is easy enough to check that $\{A \equiv_d \sim A\} \models_d B$, and so the objection would apply. Jaśkowski, presumably realizing this (but failing to give any reason) chose to define $A \equiv_d B$ as $(A \supset_d B) \wedge (B \supset_d MA)$. This avoids the problem. However, it produces a lopsided account of equivalence which is, intuitively, a symmetric operation. Moreover it results in the failure of the clearly desirable $(A \equiv_d B) \supset_d ((A \supset_d B) \wedge (B \supset_d A))$, though the converse implication holds.

It might be thought that it would be better to define $A \supset_d B$ as $MA \supset MB$. This receives the perfectly natural gloss 'if A holds discursively, B holds discursively'. So defined it would still satisfy *modus ponens* and now, moreover, discursive equivalence can be defined in the obvious way without disaster since

$$\{(MA \supset M \sim A) \wedge (M \sim A \supset MA)\} \not\models_d B.$$

However, this definition of $\supset_d$ would not have solved the other problems. For an argument exactly analogous to the previous one shows that, even as redefined, the pure $\supset_d$ fragment of the theory is the same as the pure material implication fragment of classical logic. Hence the account falls to both the irrelevancy and the Curry objections (see also footnote 157 of chapter 1).

Discursive implication whether defined in Jaśkowski's way or in our suggested way has some other undesirable features. In particular it fails several natural implication rules, e.g.

$$\{A \supset_d B\} \not\models_d {\sim}B \supset {\sim}A \qquad \{A \supset_d B, A \supset_d {\sim}B\} \not\models_d {\sim}A$$

whilst satisfying such curios as

$$\models_d (A \wedge {\sim}A) \supset_d B \qquad \models_d A \supset_d (B \vee {\sim}B).$$

For all these reasons discursive implication is an inadequate account of implication. An obvious question to raise is whether there is any definition of discursive implication which would be satisfactory. Naturally the consequences of each definition have to be looked at separately. Yet it is easy to produce one objection to *any* definition.

Let $\phi(p, q)$ be any modal sentence with two propositional parameters p, q. Then $\phi(p, q)$ is not a suitable definition of implication. For either $\not\models_d \phi(p, p)$, i.e. implicational identity fails, or $\models_d \phi(p, p)$. In this case let A be any logically true sentence. Certainly $\models_d \phi(A, A)$. Now let q be any sentential parameter not in A. Then since $\models_{S5} A \leftrightarrow q \vee {\sim}q$, $\models_d \phi(A, q \vee {\sim}q)$. Thus the implication fails the relevancy objection.

Positive-plus systems similarly fail suitability requirements for paraconsistency, as we will next explain.

3.2. Positive-plus systems, such as da Costa's main systems

Again we will start with the most accessible da Costa system, C_ω. Semantics for the full system C_ω, including its implication operator, due to Loparić,[51] take the following form: A *semivaluation* is any map ν from formulas to $\{0, 1\}$ satisfying the conditions for a zero degree da Costa evaluation (see above) plus these conditions for $\supset$:

if $\nu(A \supset B) = 0$ then $\nu(B) = 0$; if $\nu(A \supset B) = 1$ then $\nu(A) = 0$ or $\nu(B) = 1$.

A *C_ω valuation* is any semivaluation ν such that for any formula B of the $\supset (A_2 \supset A_3 \ldots A_n) \ldots)$, where A_n is not of the form $C \supset D$, if $\nu(B) = 0$ there is a semivaluation ν' such that $\nu'(A_i) = 1$, for each i such that $1 \leq i < n$, and $\nu'(A_n) = 0$. Logical truth and consequence are now defined in the usual way.

These semantics for the full C_ω are not, on their own, particularly illuminating. Hence we will depart from our usual practice of analysing logics via their semantics and approach C_ω instead by its proof theory. The standard axioms for C_ω are as follows.[52]

1. $A \supset (B \supset A)$
2. $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
3. $(A \wedge B) \supset A \quad [(A \wedge B) \supset B]$
4. $A \supset (B \supset (A \wedge B))$
5. $A \supset (A \vee B) \quad [B \supset (A \vee B)]$
6. $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
7. $A \vee {\sim}A$
8. ${\sim}{\sim}A \supset A$

The only rule of inference is *modus ponens* for $\supset$.

Those who know their intuitionism will recognize that axioms 1 and 2 are axioms for the pure calculus of intuitionistic implication and axioms 1–6 are axioms for the positive intuitionist calculus. Thus C_ω contains both these theories. In fact the axioms suggest that the implicational fragment of C_ω is exactly the pure calculus of intuitionist implication and the positive part of C_ω is exactly the positive part of intuitionistic logic. Indeed Loparić's semantics can be used to show that this suggestion is correct. C_ω is a conservative extension of positive intuitionistic logic.[53]

Thus we see that C_ω is essentially positive intuitionist logic plus the "negation" operator—really a subcontrary operator—$\sim$. As is well known, neither 7 nor 8 is intuitionistically valid, though their "opposites" ${\sim}(A \wedge {\sim}A)$ and $A \supset {\sim}{\sim}A$ are. This shows a certain symmetry between the negation of C_ω and of intuitionist logic,[54] which fits in well with the discussion of C_ω negation in 2.2. For intuitionistic negation is plausibly seen as a contrary-forming operator (rather than a contradictory-forming one): $A \wedge {\sim}A$ is logically false and $A \vee {\sim}A$ is not logically true; and the connection between modal logic and intuitionist logic suggests that the intuitionist negation of A is to be understood as something like '${\sim}A$ is provable' or 'A will never be true'; both of which are contraries of A (at least as normally understood). The "opposite" of a contrary forming operator is a sub-contrary forming operator. And this is exactly what we argued the negation of C_ω to be.

Having got all this straight, we can now see quickly that the implication operator of C_ω is inadequate. For it, like strict implication, falls to both the irrelevance objection (since intuitionist logic contains irrelevancies such as $A \supset (B \supset B)$ and $C \supset (A \supset (A \vee B)))$ and the Curry objection (since it contains $(A \supset (A \supset B)) \supset (A \supset B)$).

The transition from C_ω to C_1 (and the other C_i systems) does not make matters any better; in fact it makes them worse. For if we add to the C_ω axioms those required for the C_1 classicality operator, viz.

$$B^0 \supset ((A \supset B) \supset ((A \supset {\sim}B) \supset {\sim}A))$$

and $A^0 \wedge B^0 \supset ((A \wedge B)^0 \wedge (A \vee B)^0 \wedge (A \supset B)^0 \wedge (\sim A)^0)$ then Peirce's law $(((A \supset B) \supset A) \supset A)$ becomes provable and hence C_1 contains classical material implication.[55] In fact if we add the semantical postulate for the classicality operator to those for the Loparic semantics of C_ω we can then simplify the semantic condition for $\supset$ to the classical

$$\nu(A \supset B) = 1 \text{ iff } \nu(A) = 0 \text{ or } \nu(B) = 1$$

and the difference between valuations and semivaluations vanishes. These semantics can then be used to show that the positive fragment of C_1 is exactly the positive fragment of classical two-valued logic. Thus C_1 is exactly classical positive logic plus da Costa "negation".

The fact that C_ω (C_1) contains conservatively the positive fragment of intuitionist (classical) logic, is no accident. For one of da Costa's motivating principles for the construction of the C systems is that they 'must contain the most part of the schematic rules of... [classical logic] which do not... [interfere with their paraconsistency, or make $\sim(A \wedge \sim A)$ provable]'.[56] Thus he is committed to a very strong implication operator. This is a mistake, not only because strong implication operators are irrelevant, but because this very fact forces on da Costa his inadequate treatment of negation. For example, the fact that the theory contains the paradoxical $A \supset (B \supset A)$ means that contraposition must fail. For that (together with the transitivity of implication) leads immediately to the paraconsistently unacceptable $A \supset (\sim A \supset \sim B)$. Similarly the fact that the theory contains the paradoxical $A \supset (B \vee \sim B)$ means that either contraposition or De Morgan's law must fail. For if they held, we would have both $\sim(\sim B \vee B) \supset \sim A$ and $B \wedge \sim B \supset \sim(\sim B \vee B)$, giving the paraconsistently unacceptable $(B \wedge \sim B) \supset \sim A$. The same point may be made about the failure of contraposition in discursive logic. Moreover, the fact that $\models_d (B \wedge \sim B) \supset_d A$ forces a discursive paraconsistentist to give up the law of adjunction $\{A, B\} \models_d A \vee B$. Thus although relevance is an issue separate from paraconsistency, a cavalier attitude to relevance causes infelicities, at least, in a paraconsistent logical theory. Neither material nor intuitionist nor strict nor discursive implication is a suitable account for a paraconsistentist.

3.3. The relevant approach

All this forces us back to the third paraconsistentist approach to implication: through a relevant implication. For our present purposes we again take a broadly relevant propositional logic to be one satisfying the Anderson and Belnap variable-sharing condition.[57] Clearly, any relevant logic will avoid the paraconsistently execrable *ex falso quodlibet* and therefore will be a *prima facie* candidate for a paraconsistent logic. However, there are many approaches to relevant logic. These may be usefully classified for present purposes as follows:

$$\begin{tikzcd}[column sep=small]
 & & \text{Connexive positions} \\
 & \text{Retain Transitivity} \arrow[ur] \arrow[dr] & \\
\text{Accept Disjunctive Syllogism} \arrow[ur] \arrow[dr] & & \text{Conceptivist (Parry) systems} \\
 & \text{Reject Transitivity} \arrow[r] & \text{Sieve positions} \\
 & \text{Retain Absorption} \arrow[r] & \text{Anderson–Belnap systems} \\
\text{Reject Disjunctive Syllogism} \arrow[ur] \arrow[dr] & & \\
 & \text{Reject Absorption} \arrow[r] & \text{Depth relevant logics}
\end{tikzcd}$$

Several of these approaches are not adequate for paraconsistency (and sometimes in their own terms). So much we have already seen in the case of connexivist and conceptivist systems, which allow the spread of inconsistency.

The third approach to relevant logic insists that a suitable logic should be obtained by imposing a condition of relevance of relatedness (as a sieve) on classical truth preservation.[58] Thus $A \to B$ is supposed to hold if A materially (or strictly) implies B and R(A, B) holds where R is some suitable relation of relevance, usually taken to be some kind of meaning connection.[59] This approach we take to be fundamentally misguided, for a number of reasons. Here are some.

First, such approaches normally (and with superficial plausibility) take variable-sharing to be a sufficient criterion for relevance. If it is, then all of $(A \wedge B) \to B$, $A \to (A \vee B)$, $(A \wedge (\sim A \vee B)) \to B$ come out as relevantly valid. But these plus the transitivity of '$\to$' lead, by the usual Lewis argument to $(A \wedge \sim A) \to B$, which is clearly irrelevant. Thus, the transitivity of implication has to be given up.[60] This seems to be such a fundamental principle of implication, almost as fundamental as *modus ponens*, that it should be given up only under the most extreme of circumstances. Since there are other approaches which validate transitivity, these circumstances do not obtain.[61]

Secondly, although such approaches rather automatically avoid the irrelevance objection, they do not escape the Curry objection. For absorption, $A \to (A \to B) \to .A \to B$ is a thesis of such systems along with *modus ponens*. For $A \to (A \to B) \to .A \to B$ is a classical (or strict) thesis, and antecedent and consequent are related, i.e. $R(A \to (A \to B),\ A \to B)$, since $R(A \to B, A \to B)$ (because R is reflexive) and so relates the consequent of $A \to (A \to B)$ to $(A \to B)$. Thus such systems are quite unsuitable for major paraconsistent purposes.[62]

Thirdly the relevance relation R would seem to be, what it is usually taken to be, symmetrical. (If it is not, the nature of relevance becomes obscure.)[63] Now consider the clearly true: 'Today is Monday' implies 'Tomorrow is not Monday'. Let us write this as $A \rightarrow B$. Then since it is true, $R(A, B)$ holds, and since R is symmetrical $R(B, A)$ holds. Now suppose it is Sunday, then B is false. Thus the inference from B to A is materially truth preserving. Hence it is true that 'Tomorrow is not Monday' implies 'Today is Monday'—an obvious absurdity. A similar example can be made to work against those who wish to impose relevance on top of strict implication. '31 is an even number greater than 2' implies '31 is a composite number'. However '31 is a composite number' does not imply '31 is an even number greater than 2'. Thus relevance is not an extra condition to be tacked on, on top of truth preservation.[64] Rather relevance should be defined, as traditional logic has it, in terms of implication.[65]

A more enlightened approach to relevant logic is that of Anderson and Belnap, who *start* by trying to give an account of implication.[66] Their approach never, however, took due account of paraconsistency, and all their systems of relevant logic, namely E and T and R, fall to the Curry objection. All contain the offending rule $A \rightarrow (A \rightarrow B)/A \rightarrow B$.[67] Thus a suitable relevant paraconsistent logic can be found only in systems weaker than E, T and R, which have come to be known as 'depth relevant logics'.[68] Again there are a number of different approaches to these, and since our aim is not to give a survey of relevant logics, we will just outline one, which has a strong intuitive content.[69]

Let L be a language. Where A is a sentence of L, let [A] be the sense or objective content of A. Let $\leqslant$ be the relation of sense containment, i.e. $[A] \leqslant [B]$ iff the sense of A contains that of B (i.e. all the content of B is included in that of A). Clearly $\leqslant$ is a partial ordering. Moreover, assuming that the sense of a compound is a function of the senses of its parts, we can define the functions $\cup$, $\cap$ and $*$ thus: $[A] \cup [B] = [A \vee B]$; $[A] \cap [B] = [A \wedge B]$; $[A]^* = [\sim A]$. It can be convincingly argued that these operations turn the partial ordering of senses into a De Morgan lattice, i.e. a distributive lattice for which $\cup$ is the join, $\cap$ is the meet, and $*$ is an involution, i.e. a function such that $a^{**} = a$ and if $a \leqslant b$ then $b^* \leqslant a^*$. Thus, for example, the sense of $A \wedge B$ contains both the sense of A and that of B. Moreover anything that contains both senses also contains that of $A \wedge B$. Thus a De Morgan lattice can be seen as a lattice of senses.

Now an algebra of senses allows us to define entailment in a very natural way. For it is plausible to suppose, as many have done, that an entailment is true precisely if the sense of the antecedent contains that of the consequent. Thus $A \rightarrow B$ is true iff $[A] \leqslant [B]$. Formally, if T is the set of senses of true sentences, $[A \rightarrow B] \in T$ iff $[A] \leqslant [B]$. It is also reasonable to suppose that T is at least a prime filter on the lattice, i.e. that $a \cap b \in T$ iff $a \in T$ and $b \in T$;

$a \cup b \in T$ iff $a \in T$ or $b \in T$; and if $a \in T$ and $a \Rightarrow b \in T$, $b \in T$ (where $[A] \Rightarrow [B]$ is $[A \rightarrow B]$).

Further details of the lattice and truth filter T are more negotiable, but it is already clear that these semantics show all the following to be logically true

$$A \rightarrow A, \; A \rightarrow \sim\sim A, \; \sim\sim A \rightarrow A, \; A \wedge B \rightarrow A, \; A \rightarrow A \vee B$$

and the following inferences to be truth preserving

$$A \rightarrow B, B \rightarrow C / A \rightarrow C \qquad A \rightarrow B / \sim B \rightarrow \sim A$$

$$A \rightarrow B, A \rightarrow C / A \rightarrow B \wedge C \qquad A \rightarrow C, B \rightarrow C / A \vee B \rightarrow C.$$

$$A, A \rightarrow B / B$$

which is what we would expect of an entailment operator $\rightarrow$. Hence it is clear that these details provide the basis of a semantics for entailment.

It may not be clear how these semantics relate to those for the zero degree case we discussed in 2.3. The connection is this:—[70] Suppose we define a map ν from zero degree formulas to $\{0, 1\}$ as follows: $1 \in \nu(A)$ iff $A \in T$; and $0 \in \nu(A)$ iff $\sim A \in T$. Then ν is a zero degree valuation of the kind specified in 2.3. To be more precise the semantics as specified make ν a map to $V \cup \{\phi\}$, thus allowing for truth value gaps (see fn.43). The further condition: $a \in T$ or $a^* \in T$ makes ν a map to V. Thus these semantics subsume the zero degree semantics and extend them to higher degrees.

An implication based on these semantics is very satisfactory for paraconsistent purposes and suffers from none of the problems of the implications of the previous two approaches: it is relevant; the Curry-paradox generating $A \rightarrow (A \rightarrow B) / A \rightarrow B$ fails; negation has the right properties (contraposition, De Morgan, double negation), etc. Moreover, as we shall see subsequently, naive set theory and semantics based on this kind of relevant logic, though they may be inconsistent, are provably non-trivial.[71] Hence we conclude that this is the most suitable approach to implication for paraconsistent purposes, and that, more generally, the relevant approach to paraconsistency is the most satisfactory one.

Notes

[1] Berkeley, 1734. Further details of the story can be found in Boyer, 1949.
[2] See Lakatos, 1970, §3(c2).
[3] See e.g., Feyerabend, 1978, IV.
[4] See Priest, 1980.
[5] See the Introductions to Parts Three and Four of this volume.
[6] For Riggs and Palmer, see 115 N.Y. 506, 22 N.E. 188 (1889) and Dworkin, 1977, p. 23. On the Proclamation of Emancipation, see Hook, 1962, p. 28. The section

in which this point is made contains a discussion of several other inconsistencies in the American Bill of Rights.

[7] For many examples of inconsistent sets of beliefs, see R. and V. Routley, 1975. For elaboration of the computer example, see N.D. Belnap Jr., 1977.

[8] See Hegel, 1812, vol. 1, Bk. 2, Ch. 2, §C.

[9] See chapter II, pp. 77f. where also further discussion of Hegel and Zeno, may be found.

[10] This example is much further developed in the introduction to Part Four, where two other plausible examples of true contradictions are given, e.g. those posed by certain impossible objects supplied by Meinong's theory of objects.

[11] For further discussion see Priest, 1980.

[12] This is one of the *many* reasons for doubting the adequacy of such positions.

[13] Details of these logics, and proofs that they fail to meet paraconsistency requirements, are given in chapter 2 of R. Routley *et al.*, 1982. See especially p. 93 and p. 101. Connexive logics are flawed paraconsistently because they admit the inference, $\{A, \sim A\} \models B$. The point, argued syntactically (p. 93), may be reargued semantically, as follows: Since in connexive logics A and $\sim A$ cancel one another, A and $\sim A$ are never designated together, and $A \wedge \sim A$ is always non-designated. Thus both $\{A, \sim A\} \models B$ and $\{A \wedge \sim A\} \models B$ hold (on designation-preserving accounts), and connexive logics are not paraconsistent. Quite apart from this, connexive logics would be pretty useless for genuine dialectic purposes. For, as with one of the leads Wittgenstein pursued (discussed in an Introduction to Part One of the book), contradictions stop things, so undercutting much legitimate reasoning. The argument that Parry logics and the like (e.g. Zinov'ev's system) fail to be paraconsistent, also given syntactically (p. 101), may likewise be reworked semantically. For on the so far received semantics for these systems, A and $\sim A$ are never designated together, and $A \wedge \sim A$ is not designated.

[14] See, for instance, the account given of modal logics in RLR, chapter 1, upon which the argument in the text depends.

[15] Jaśkowski, 1948.

[16] See e.g., da Costa and Dubikajtis, 1968; Kotas and da Costa, 1978.

[17] See Rescher and Brandom, 1980. A more sophisticated form is to be found in Schotch and Jennings, 1987. Most of our criticisms apply to them too.

[18] The idea can be traced back at least to the Jains. It reappears in classical Greek thought: see chapter I, this vol.

[19] Why non-recursive semantics are philosophically unsatisfactory is explained below. The truth of the claim is easily seen from the fact that we may have $\mathcal{M} \models_d B$ and $\mathcal{M} \models_d C$ whilst $\mathcal{M} \models_d A \wedge B$ but $\mathcal{M} \not\models_d A \wedge C$. Thus there is no such condition ψ in the standard (extensional) set-theoretic metalanguage of modal logic.

[20] See e.g. da Costa and Dubikajtis, 1968.

[21] The charge is serious since this is one of the prime motivations of paraconsistency, and one moreover cited by Jaśkowski, 1948, p. 143 ff.

[22] Rescher and Brandom, 1980, ch. 10. An analogous proposal, which drastically weakens set theory, was earlier investigated by Gilmore, 1973, in his partial set theory.

[23] It remains an important game, which with paraconsistent theory can be tackled in a thoroughly systematic way for the first time. For the theory to be partitioned can now be formalized non-trivially.

[24] Jennings and Schotch's account avoids this problem.

[25] Jaśkowski, 1948, p. 145.

[26] A detailed critique of modal approaches to non-modal matters (such as deducibility, entailment, and paraconsistency) is included in RLR, chapter 1, especially 1.6. This is not to imply that no intensional functors violate adjunction: some certainly do, but their proper treatment is not a modal one.

[27] He has written a number of papers on the subject. The best place to start is with da Costa, 1974. The C systems are not the only paraconsistent systems due to da Costa: he is also jointly responsible with Arruda for the basic P systems; see the first Introduction to Part One of the book. A rather different positive approach is that of Peña 1989.

[28] See da Costa and Alves, 1976; Loparić, 1977.

[29] In fact it can be shown that under the very weak condition that no non-theorem has the same sense as any theorem, the slightly stronger da Costa system C_1 has no non-trivial recursive sense-semantics. For suppose it did. Let S be the set of senses (subsets of possible worlds or whatever) and let $\cong$ be the relation which holds between two formulas if they have the same sense. Then the set of formulas factored by $\cong$ would be a non-trivial quotient algebra for C_1. But there is no such algebra for C_1, as Mortensen, 1980, shows.

[30] da Costa, 1974, p. 498.

[31] At least for formal logic. See Hegel, 1830, note on §20, p. 32.

[32] The da Costa formulation is the equivalent: if $\nu(A \supset B) = \nu(A \supset \sim B) = 1$ then $\nu(A) = 0$.

[33] See da Costa, 1974, theorem 18.

[34] See da Costa, 1974, p. 505.

[35] See, in particular, Arruda, 1982. See also Arruda and Batens, 1982.

[36] See, e.g., Priest, 1980a, and Routley, 1979.

[37] This is the way of Priest, 1979, though we formulate it in the manner of Dunn, 1976. For an alternative approach to the semantics of zero degree relevant logic, see Routley and Routley, 1972. These and other approaches are elaborated and discussed in RLR 3.1 and 3.2.

[38] See Priest, 1979, Theorem III.8.

[39] All the relations (κ) of 2.2 hold. When we come to implication we will see that those of (λ) hold too.

[40] Further discussion is to be found in Priest, 1989; and Routley 1978. A detailed discussion may be found in RLR.

[41] See Anderson & Belnap, 1975, §16.3.

[42] Priest, 1989.

[43] As an example of how the semantics may be modified, consider those given in this section. We merely extend V to include the empty set (as in Dunn, 1976). All else remains the same. It is easy to check that the main effect of this is to ensure that $\nvDash A \vee \sim A$ and in fact that there are no theorems at all! The consequence relation is of course still non-trivial. A similar phenomenon occurs when the semantics are extended to allow for an implication operator. In this case although the logic now has theorems there are no purely extensional ones, i.e. ones containing only $\wedge$, $\vee$ and $\sim$. The holding or failing of purely extensional theorems is of little technical relevance to paraconsistency: it can be done either way. However, for reasons indicated in the text we think that a semantics which does not validate the laws of excluded middle and non-contradiction opens itself to the charge that it has not given a semantic account of *negation.* A discussion of truth value gaps in the context of Meinong's theory can be found in EMJB §1.2.

[44] See Jaśkowski, 1948, p. 147.

[45] For example see Anderson and Belnap, 1975, Routley and others, 1982, Routley and Norman, 1988.

[46] For example, Ackermann's original *Strenge Implikation* in 1956, which uses the disjunctive syllogism in rule form.

[47] See Anderson and Belnap, 1975, p. 32f.

[48] See Curry, 1942. Different versions are given in Priest, 1979, and Meyer, Routley and Dunn, 1979.

[49] See Jaśkowski, 1948, p. 150.
[50] Jaśkowski, 1948, Theorems 1, 3.
[51] See Loparić, 1977.
[52] See da Costa, 1974.
[53] See Loparić, 1977, p. 838.
[54] Giving some warrant to the label 'anti-intuitionistic' sometimes applied to logics like the C systems.
[55] See da Costa and Guillaume, 1965.
[56] da Costa, 1974, p. 498.
[57] As before a purely propositional logic is (weakly) relevant iff there is no theorem of the form $A \rightarrow B$, where A and B have no propositional variable in common.
[58] For examples of this approach see Epstein, 1979, Copeland, 1980, Epstein and Szcerba, 1979.
[59] In fact various conditions of relevance (beginning with simple syntactical requirements such as variable overlap or inclusion) can be imposed on a wide variety of logics. The more general procedure enables not merely various non-transitive logics, but also certain Parry and depth relevant logics, to be represented as imposing a filter on classical or modal logics.
[60] Non-transitive theories of implication, especially a feature of the Cambridge tradition go back at least to Strode in the Middle Ages. Typically non-transitivity resulted by adding further requirements to material or strict implication, often epistemic requirements concerning ways of coming to know the truth of the implication. The initially unspecified relation R, of relational implication, is simply the latest, and in some respects the crudest, of these attempts to filter out the bad guys, but like the usual sieves fails abysmally in this task, admitting Disjunctive Syllogism and eliminating Transitivity. (On the respective virtues of these principles, see RLR, chapter 2.) Another subtler way of instituting the filter, with a rather similar outcome, however, is that of Tennant, 1984. The resulting relevant system satisfies Disjunctive Syllogism, at the expense of faulting transitivity. Thus, the system is unsuitable for the representation of many inconsistent theories which are closed under logical consequence. Furthermore the system is open to the Curry objection.
[61] The case for Transitivity and against its rejection is developed at length in RLR, especially in the initial parts of chapter 2.
[62] Some of the leading exponents of relatedness logics in fact seem oblivious to the fact that an important role for deductive—and also inductive—logic is reasoning from, or in the presence of, inconsistency. Otherwise, presumably, Woods and Walton would not have begun their text on fallacies with the following unclassified fallacy: "Unlike deductive or inductive logic, the plausible model of argument allows us to deal with cases where we are confronted with contradictions" (1982, p. vii).
[63] A relation R for which symmetry is not required is considered by Epstein, 1979. But it remains uninterpreted, other than merely formally, whereas the symmetric relation can be interpreted in terms of shared topics or subject matter. Naturally there are non-symmetric relations of interest here, e.g. the transitive relation of variable inclusion, important for some Parry logics. But these are not, or not *merely*, relevance relations. (For a discussion of Parry logics see RLR.)
[64] There is much else wrong with the tack-on idea, as for instance implemented in relational logic. For example, it validates Ackermann fallacies, such as $A \rightarrow . (A \rightarrow B) \rightarrow B$, which are also readily counterexampled as implicational and conditional principles. It also fouls up expected and legitimate substitution conditions, such as inter-replacement of co-entailments, and can interfere with substitution on variables.

[65] Thus, for example, A is (implicationally) relevant to B iff A is not independent of B, i.e. iff A is either a superimplicant, subimplicant, equivalent, contrary, subcontrary or contradictory of B, i.e. iff one of $A\to B$, $B\to A$, $A\to\sim B$, $\sim A\to B$ holds.

[66] See Anderson and Belnap, 1975.

[67] Even a version of R without the absorption axioms has a version of the Curry paradox: see Slaney, 1989. Further arguments against the Anderson and Belnap systems and in favour of depth relevant logics can be found in Priest and Routley, 1982.

[68] For the term see Brady, 1982.

[69] This is given in Priest, 1980a. It is similar to the range semantics of Routley and Routley, 1972, with the restriction to first degree wff lifted, something in effect carried out—and done explicitly for the dual, content semantics—in EMJB, Appendix, where further details may be found. Moreover these semantics are readily embedded in more familiar semantics for relevant logics; see e.g. RLR, chapter 3.

[70] See Priest, 1980a, Appendix.

[71] See Brady, 1989.

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